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Book Review: *Scaling Limits of Interacting Particle Systems*

Scaling Limits of Interacting Particle Systems. Claude Kipnis and Claudio Landim, Springer, New York.

To call upon the authors:

> The problem we address in this book is to justify rigorously a method often used by physicists to establish the partial differential equations that describe the evolution of the thermodynamic characteristics of a fluid or a gas... The equilibrium states of the system are characterized by a small number of macroscopic quantities $\mathbf{p}$ called the thermodynamic characteristics, e.g., the density... Assume the gas is confined in a volume V, we may investigate the evolution of the system out of equilibrium. For each point u in V, V_u denotes a small neighborhood of u, that is, small compared to the total volume V, but large enough if compared to the intermolecular distance, in order to assume that each neighborhood contains an infinite number of particles. Due to the "strong" interaction among the molecules, the system reaches immediately a local equilibrium state, i.e., in each neighborhood V_u the system is close to an equilibrium state characterized by the parameter $\mathbf{p}(u)$. As the system evolves, we expect to observe at a later time t in V_u a state close to a new equilibrium state characterized by a parameter $\mathbf{p}(t, u)$, which evolves smoothly in time according to a partial differential equation, called the hydrodynamic equation... In this book we present general methods that permit to deduce the hydrodynamic equations of the thermodynamic characteristics of infinite systems assuming that the underlying microscopic dynamics is stochastic, i.e., to deduce the macroscopic behavior of the system from the microscopic interaction among particles. The microscopic systems will be the so-called interacting particle systems.

Interacting particle systems are continuous time Markov processes introduced by F. Spitzer in 1970 (Interaction of Markov processes, *Adv. Math.* **5**, 246–290), to model physical systems with probabilistic tools. The real systems being too intricate, more tractable examples were defined as toy models, such as exclusion and zero range processes. Both describe particles jumping on the sites of a lattice, according to some interaction between them: For exclusion, there cannot be more than κ particles per site ($\kappa = 1$ corresponds to simple exclusion); for zero range, a particle leaves a

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site with a rate depending on the number of particles present on that site. These processes have as (unique) conserved quantity the total number of particles. Their equilibrium states are product probability measures, parametrized by the mean density of particles per site. Existence, main properties and equilibrium behavior of most interacting particle systems are gathered in the book [L], *Interacting Particle Systems*, by T. M. Liggett (1985, Springer).

The study of non-equilibrium behavior was initiated in the eighties. However, the available tools (given in [L]) were insufficient. The basic innovating paper was [GPV] (Guo, M. Z., Papanicolaou, G. C., Varadhan, S. R. S.: Nonlinear Diffusion Limit for a System with Nearest Neighbor Interactions. *Comm. Math. Phys.* **118**, 31–59) in 1988, followed in 1989 by [KOV] (Kipnis, C., Olla, S., Varadhan, S. R. S.: Hydrodynamics and Large Deviations for Simple Exclusion Processes. *Comm. Pure Appl. Math.* **XVII**, 115–137). They introduced entropy techniques and large deviations to derive hydrodynamic equations. These two papers were initiating a new theory, implemented mostly around S. R. S. Varadhan and H. T. Yau. Indeed, a lot of new ideas spread out quite fast, appearing in long and technical papers; each one elaborated some new technology on a specific process, somehow hiding the key arguments behind the handled example. Those papers contained many ideas coming from functional analysis or PDE's. They often referred to one another. Moreover, the previous books mentioning the subject appeared in 1991, thus could only present [GPV]: *Mathematical Methods for Hydrodynamic Limits*, by A. De Masi, E. Presutti (LN in Math. 1501, Springer); *Large Scale Dynamics of Interacting Particles*, by H. Spohn (Springer).

The book by Claude Kipnis and Claudio Landim gathers the results of this new theory in a very pedagogical way. It explains the derivation of hydrodynamic equations and natural extensions, such as large deviations and fluctuations. The involved techniques are progressively introduced and detailed (from reversible to non reversible cases, from gradient to non-gradient systems), on generic examples, chosen to avoid non basic intricacies: Exclusion and zero range processes. The book is self-contained, all the necessary tools being presented in the appendices; it has an exhaustive reference list. Its natural readers are probabilists and mathematical physicists.

To give an informal description of the subject, let $(\eta_t)_{t\geqslant 0}$ be the studied process, L_N and S_t^N its generator and semi-group. Particles evolve on the discrete torus $\mathbf{T}_N^d=\{0,..., N-1\}^d$ $(d\geqslant 1)$, $\eta_t(x)$ is the number of particles on site x at time t. The invariant product measure of parameter α is denoted by ν_α. Its explicit knowledge is essential. Then, for a smooth function $\rho_0\colon \mathbf{T}^d\to\mathbf{R}_+$, $\nu^N_{\rho_0(\cdot)}$ is the product measure with mean density of particles $\rho_0(x/N)$ at site x.

To go from the *microscopic scale* $\mathbf{T}_N^d$ to a *macroscopic* one, the torus $\mathbf{T}^d = [0, 1)^d$, where the hydrodynamic equation will take place, time and space have to be rescaled: A site x in $\mathbf{T}_N^d$ corresponds to a point x/N in $\mathbf{T}^d$, a macroscopic time t to a microscopic time $t\theta(N)$ ($\theta(N)$ is equal to N or N^2, depending on the characteristics of the process).

A sequence of probability measures $(\mu^N)_{N \geqslant 1}$ is a *local equilibrium of profile* ρ_0 if $\lim_{N\to\infty} \tau_{[uN]}\mu^N = \nu_{\rho_0(u)}$ for all continuity points u of $\rho_0(\cdot)$ (it means weak convergence; $\tau_{(\cdot)}$ is the space shift). This *local equilibrium is conserved* if there exists a function $\rho: \mathbf{R}_+ \times \mathbf{T}^d \to \mathbf{R}_+$ such that $\lim_{N\to\infty} S^N_{t\theta(N)} \tau_{[uN]}\mu^N = \nu_{\rho(t,u)}$ for all $t \geqslant 0$ and all continuity points u of $\rho(t, \cdot)$. There, $\rho(t, \cdot)$ solves the *hydrodynamic equation*, i.e., a Cauchy problem with initial condition $\rho_0(\cdot)$.

Instead of conservation of local equilibrium, a weaker formulation is the convergence in probability of the *empirical measure* $\pi_t^N(du) := N^{-d} \sum_{x\in\mathbf{T}_N^d} \eta_{t\theta(N)}(x)\, \delta_{x/N}$ to $\rho(t,u)\, du$, i.e., the convergence of the law Q^N of π_t^N to the Dirac measure concentrated on $\rho(t, \cdot)$.

To proceed, for a fixed smooth function $H: \mathbf{T}^d \to \mathbf{R}$, introduce the martingale

$$M_t^H = <\pi_t^N, H> - <\pi_0^N, H> - \theta(N) \int_0^t L_N <\pi_s^N, H>\, ds$$

with $<\pi_t^N, H> := N^{-d} \sum_{x\in\mathbf{T}_N^d} H(x/N)\, \eta_{t\theta(N)}(x)$; then one has to prove that the sequence (Q^N) is tight, and take a converging subsequence, to some Q^*. By Doob's inequality, the martingale goes to 0; by initial local equilibrium $<\pi_0^N, H>$ converges to $\int_{\mathbf{T}^d} H(u)\, \rho_0(u)\, du$. The difficult part is to replace $L_N <\pi_s^N, H>$ by a function of $<\pi_s^N, H>$, to obtain a *closed equation* for the empirical measure: Since $L_N\eta(x)$ is a difference of *currents*, a first integration by parts in the integral is possible, where H is replaced by its discrete derivative, $L_N\eta_s(x)$ by a current, and a new N^{-1} appears. To get rid of the $\theta(N)$ in front of the integral, if $\theta(N) = N^2$, another step is required: Whether the current writes $h - \tau_1 h$ for some cylinder function h, the system is called *gradient*, and a second integration by parts gives another N^{-1}; or the system is *non-gradient*, and a whole machinery is required to get the correct order in N. The program is achieved by the so-called *replacement lemma*, and the hydrodynamic equation is exhibited. To finish the proof, it remains to get some uniqueness result on the weak solutions of this equation, and also uniqueness of the limit point Q^*.

The book is organized in the following way:

Chapter 1 introduces the notions of hydrodynamic limit, conservation of local equilibrium, *equivalence of ensembles*, on a non interacting system: Independent random walks.

Chapter 2 introduces simple exclusion, zero range and generalized exclusion processes, with their needed properties (invariant and translation invariant measures, attractivity), as well as the topologies of their respective state spaces, and of the associated probability spaces (e.g., characterizations of weak convergence).

Chapter 3 presents weak formulations of local equilibrium (that will be derived in the absence of some additional property of the studied process, as attractivity for instance).

Chapter 4 details, for the simplest interacting system, nearest neighbor symmetric simple exclusion, all the steps to obtain the hydrodynamic equation: Tightness of (Q^N), uniqueness of its limit points, uniqueness of weak solutions to the hydrodynamic equation.

At the end of each chapter, besides comments and references, related topics linked to the questions met within the chapter are developed, from their exposition up to the last results obtained, and to open problems. Here are mentioned: Tracer particles and self diffusion, Einstein relations, propagation of chaos.

Chapter 5 deals with nearest neighbor symmetric zero range, taken as prototype of a *reversible gradient system*. Here is detailed the replacement lemma, composed of the *one-block and two-blocks estimates*; both are based on an analysis of the *entropy production*: This is the [GPV] method. This chapter ends with H_{-1}-norm estimates, to precise the domain of Q^*, and an energy estimate for the trajectories $\rho(t, u)$, which induces a simple proof of uniqueness for solutions of the hydrodynamic equation in dimension 1.

The last section speaks of systems in contact with stochastic reservoirs.

Chapter 6 explains the *relative entropy method*, valid for systems whose hydrodynamic equation admits smooth solutions. This method implies uniqueness of those smooth solutions.

The last section mentions Euler equations, Cahn–Hilliard equations, reaction-diffusion equations, Stefan problems, Carleman and Broadwell equation, Boltzmann equations, degenerate diffusions, interface motion, motion by mean curvature and Ising models with long range interactions, interface dynamics and reaction-diffusion equations, interacting diffusions, weakly interacting diffusions.

Chapter 7 details hydrodynamics of nearest neighbor symmetric generalized exclusion, the simplest *reversible non-gradient system*. It involves the replacement of currents by gradients, an ad hoc *integration by parts formula*, estimates of variance and diffusion coefficient (known only in dimension 1); all this is based on a sharp estimate of the spectral gap for the generator of the process restricted to finite cubes, and uses the concept of *closed and exact forms*.

The last section speaks of Navier–Stokes equations: The incompressible limit, first order correction to the hydrodynamic equation, long time behavior.

Chapter 8 derives the hydrodynamic limit for an example of asymmetric attractive process, a zero range: Young measures are introduced to obtain an entropy inequality at a microscopic level, leading, by taking limits, to Kružkov entropy inequality, which characterizes the entropy solution to the hydrodynamic equation. This method requires the initial measure to be product.

Last section deals with extensions (to infinite volume, to non-product initial states, to spatially inhomogeneous processes, to continuous spin systems, to random rates), long time behavior of weakly asymmetric processes, tracer particles, central limit theorem for a tagged particle, propagation of chaos, large deviations, fluctuations of the empirical measure.

Chapter 9 continues the preceding example, to get conservation of local equilibrium via a one-block estimate without time average. This method applies to any attractive system.

The last section speaks of the microscopic structure of the shock, asymptotics of a second class particle, behavior at discontinuity points of the profile or dynamical phase transition, stationary measures of asymmetric systems. All these questions are developed in T. M. Liggett's last book: *Stochastic Interacting Systems*: *Contact*, *Voter and Exclusion Processes* (1999, Springer).

Chapter 10 comes back to symmetric simple exclusion starting from an equilibrium product space, to derive large deviations from the hydrodynamic limit; it involves an auxiliary weakly asymmetric exclusion process, and *superexponential estimates*: This is the [KOV] method.

Last section mentions extensions (to infinite volume, to non-conservative systems, to non-gradient systems), Onsager–Machlup time reversal relation, metastability, exit points from a basin of attraction, escape from unstable equilibrium points, rare events, large deviations of asymmetric models.

Chapter 11 investigates the fluctuations of the empirical measure around the hydrodynamic limit starting from an equilibrium state, for an example of reversible dynamics, a nearest neighbor symmetric zero range; it goes through the *Boltzmann–Gibbs principle*, and relies on Holley and Stroock's theory of generalized Ornstein–Uhlenbeck processes.

Appendix 1 gives the needed results for Markov chains on a countable state space.

Appendix 2 deals with equivalence of ensembles and the local limit theorem, large deviations general results (for Chap. 10), PDE's results.

Appendix 3 explains non-gradient tools (for Chap. 7): Spectral gap, closed and exact forms.

To finish, let us mention some progresses made since the book appeared: Refinements in the study of relative entropy (Kosygina E.: The behavior of the specific entropy in the hydrodynamic scaling limit, preprint, 1999); fluctuations of the asymmetric simple exclusion process, on the one hand in a one-dimensional non-equilibrium totally asymmetric case (Johansson K.: Shape fluctuations and random matrices, preprint, 1999), on the other hand equilibrium fluctuations in dimensions strictly larger than 2 (Chang, C. C., Landim, C., Olla, S.: Equilibrium fluctuations of asymmetric exclusion processes in dimension $d \geqslant 3$, preprint, 1999).

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